Lesson 31 - Lagrange Multipliers

I. Picture

II Technique

III. Examples

Announcements

① No MA 16020 next week M 11/20, W 11/22, F 11/24

② No office hours next week. If you need to meet this week on M 12:30-1:00 PM (11/13), W 11:30-12:00 (11/15), or F 10:30-12:00 (11/17), email Dr. Robbins

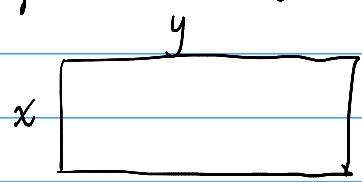
③ Quiz Friday - Lagrange Multipliers

I. Picture

We want to find maximum area that can be enclosed by a rectangular fence that has perimeter of 100 ft.

max $A = xy$

Subject to: $2x + 2y = 100$
(S.t.)



Calc I techniques can solve this

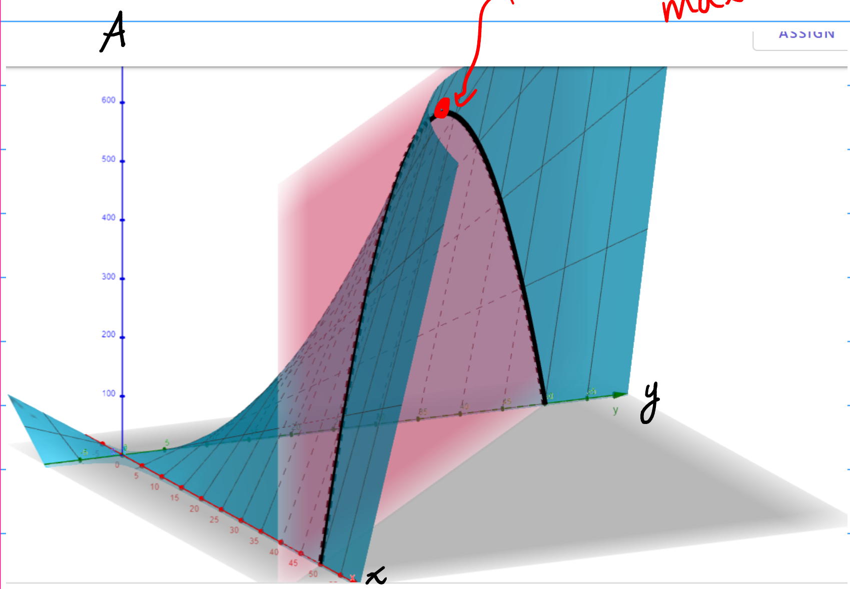
Because $A(x,y) = xy$ is a fctn of 2 vars, we have new techniques for solving thi

$$\max \quad A(x,y) = xy$$ objective function

Subject to: $2x + 2y = 100$ constraint equation

Pictures from Geogebra:

where you attain max A value



Blue: $A = xy$

pink: $2x + 2y = 100$

See gradient file on our website

One way to encode that two functions have the same tan line/normal lines @ (a,b) is to say that

$$f_x(a,b) = \lambda g_x(a,b)$$

and

$$f_y(a,b) = \lambda g_y(a,b)$$

for some scalar (constant) λ

II. Technique

objective function

To solve max or min $z = f(x,y)$

subject to $g(x,y) = 0$

constraint function set $= 0$

① Solve the system of equations

$$\begin{cases} f_x(x,y) = \lambda g_x(x,y) \\ f_y(x,y) = \lambda g_y(x,y) \\ g(x,y) = 0 \end{cases}$$

for (x,y) and λ

② Check to make sure your max or min occurs @ (x,y) by substituting any other point on $g(x,y)=0$ into $z=f(x,y)$.

III. Examples

$$\max\ A(x,y) = \overbrace{xy}^{f(x,y)}$$

$$\text{s.t.}\quad 2x+2y = 100 \longrightarrow \underbrace{2x+2y-100}_{g(x,y)} = 0$$

$$\begin{cases} A_x(x,y) = \lambda g_x(x,y) \\ A_y(x,y) = \lambda g_y(x,y) \\ g(x,y) = 0 \end{cases} \rightarrow \begin{cases} y = \lambda \cdot 2 & ① \\ x = \lambda \cdot 2 & ② \\ 2x+2y-100 = 0 & ③ \end{cases}$$

① and ② $\longrightarrow$ ③

$$2(2\lambda) + 2(2\lambda) - 100 = 0$$
$$8\lambda = 100$$
$$\lambda = 12.5$$

$\lambda = 12.5 \rightarrow$ ① and ②

$$y = (12.5)\cdot 2 = 25$$
$$x = (12.5)\cdot 2 = 25$$

Let's hope max is @ $(x,y) = (25,25)$

$A(25,25) = (25)(25) = 625$

Pick any other point on $2x+2y-100=0$

$2x+2y=100$

Often easiest to pick a pt where $x=0$ or $y=0$

I choose $x=0$ $\qquad 2(0)+2y=100$

$y=50$

$(0,50) \rightarrow A(0,50)=0\cdot 50 = 0 < 625$

So max area is $625\ ft^2$ when the fence is a 25×25 square.

$\boxed{EX}$ (Openstax, Calc III, Ex 4.4.2)

min $f(x,y) = x^2+4y^2-2x+8y$

s.t. $x+2y=7$

$g(x,y) = x+2y-7 = 0$

$$\begin{cases} f_x(x,y) = \lambda g_x(x,y) \\ f_y(x,y) = \lambda g_y(x,y) \\ g(x,y)=0 \end{cases} \rightarrow \begin{cases} 2x-2 = \lambda\cdot 1 & (1) \\ 8y+8 = \lambda\cdot 2 & (2) \\ x+2y=7 & (3) \end{cases}$$

(1) $\lambda = 2x-2 \longrightarrow$ (2) $8y+8 = (2x-2)\cdot 2$

$8y+8 = 4x-4$

$4x = 8y+12$

$x = 2y+3$ (4)

$\downarrow$ (3)

$2y+3+2y=7$

$4y=4$

$y=1 \longrightarrow$ (4) $x = 2(1)+3 = 5$

$(5,1)$ $\quad f(5,1) = 5^2+4(1)^2-2(5)+8(1) = 27$

Any other point on $x+2y=7$

I choose $y=0 \Rightarrow x+0=7 \Rightarrow x=7$

$(7,0)$ $\quad f(7,0)=7^2+4(0)^2-2(7)+8(0)=35>27$

Minimum of 27 @ $(5,1)$

$\boxed{\text{EX}}$ (openstax, Calc III, §4.8 #358)

Find max and min of

$$f(x,y)=x^2y$$

subject to $\underbrace{x^2+2y^2-6}_{g(x,y)}=0$

$$\begin{cases} 2xy=\lambda 2x & (1) \\ x^2=\lambda 4y & (2) \\ x^2+2y^2=6 & (3) \end{cases}$$

Let's look @ (1).

$$2xy=\lambda 2x$$

You cannot divide by x.

$$2xy-2x\lambda=0$$
$$2x(y-\lambda)=0$$

$x=0$ or $y=\lambda$

Case 1: $x=0$

$\rightarrow (3)$

$$0^2+2y^2=6$$
$$y^2=3$$
$$y=\pm\sqrt{3}$$

Points to consider

$(0,\sqrt{3})$

$(0,-\sqrt{3})$

Case 2: $y=\lambda$

(2) $x^2=\lambda 4y$

$\Rightarrow x^2=y4y$

$x^2=4y^2$

(3) $x^2+2y^2=6$

$$4y^2+2y^2=6$$
$$6y^2=6$$
$$y^2=1$$
$$y=\pm 1$$

→ 3 $y=1$

$$x^2+2(1)^2=6$$
$$x^2=4$$
$$x=\pm 2$$

Points: $(2,1)$ $(-2,1)$

$y=-1$

$$x^2+2(-1)^2=6$$
$$x^2=4$$
$$x=\pm 2$$

Points: $(2,-1)$ $(-2,-1)$

Point	$f(x,y)=x^2y$
$(0,\sqrt{3})$	0
$(0,-\sqrt{3})$	0
$(2,1)$	4
$(2,-1)$	-4
$(-2,1)$	4
$(-2,-1)$	-4

No need to check more pts on $g(x,y)=0$ because we already have at least 2 values of $f(x,y)$

max of 4 @ $(2,1)$ and $(-2,1)$
min of -4 @ $(2,-1)$ and $(-2,-1)$